Mean-Variance Analysis of the Newsvendor Problem with Price-Dependent, Isoelastic Demand

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Abstract

We present a mean-variance analysis of the single-product, single-period newsvendor problem with two decision variables, price and stock quantity. The demand is price-dependent, multiplicative, and isoelastic, and the product sold is price-elastic. The main goal of this paper is to completely characterize a framework where most risk sensitivities can be studied and find conditions under which the unimodality of this mean-variance performance measure is guaranteed. We aim at presenting these conditions in terms of a metric that can capture managerial attention. To this end we use the lost sales rate (LSR) elasticity, as it relates directly to the retailer's level of service. The main contribution of this paper is that, with very few and mild assumptions, we complement currently existing results for the uniqueness of the solution in the presence of price-inelastic products by extending the applicability of the LSR elasticity to the problem with price-elastic goods and assess its unimodality when the decision maker is risk-neutral or risk-sensitive. Finally, we compare the results under this model with others previously obtained for additive demand models, and we emphasize the reasons that make it difficult to attain similar results with an isoelastic demand model.

Keywords: Inventory, Pricing, Newsvendor, Risk analysis, Revenue Management

1. Introduction

The newsvendor problem has been subject to extensive research since it first appeared at the end of the *XIX* century. It is a conceptually simple, yet powerful model to set optimal policies in order to mini-

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mize replenishment costs (when setting only order quantities) or maximize profits (when setting both order quantities and selling prices) in the face of uncertain demand. The amount of research conducted on this model and its applications is so vast that a simple search in Google Scholar under the keyword *"newsvendor"* will return about $8,000$ entries between 2013 and 2018. However, this introduction aims at referring to the problem herein exposed and, in view of the size of the body of literature, we may miss mentioning some significant works inadvertently. For a thorough review of this problem in the context of supply chain management, we refer the readers to Chan et al. (2004), Chiu and Choi (2016), Khouja (1999), Qin et al. (2011).

In terms of the number of variables in the problem, the simplest newsvendor models consider only stock optimization. In the last few years, there has also been an important number of publications that introduce this problem under different risk considerations. Normally, the newsvendor is presented as a risk-averse individual that aims at maximizing profit or minimizing replenishment costs. The literature offers distinct methods to model this risk aversion, each of them having advantages and drawbacks. Examples of approaches under risk for determining optimal stock quantities include the well-known utility function framework as in Lau (1980), Bouakiz and Sobel (1992), Eeckhoudt et al. (1995), mean-variance analysis as in Lau (1980), Chen and Federgruen (2000), value at risk (Özler et al., 2009), and conditional value at risk (Ahmed et al., 2007, Choi and Ruszczyński, 2008, Jammernegg and Kischka, 2007).

Another important feature of this problem is the demand for the goods. Often, the demand is assumed to take on values according to a statistical distribution. In other cases, the stochasticity of the demand is only given by a random perturbation of a function. This function usually depends on a very reduced number of variables. It is common, for example, to consider that the demand is represented as a decreasing function of the price and modified according to a random variable (Karlin and Carr, 1962, Mills, 1959, Federgruen and Heching, 1999, Petruzzi and Dada, 1999).

The problem proposed in this paper is concerned with a single-stage, single-period newsvendor that aims at setting optimally both price and order quantity. We use a well-known functional form of the demand: the so-called multiplicative, isoelastic demand model. The isoelastic function only produces a change in the scale of the demand distribution and, for every context, it remains an open question if a given commodity exhibits a change in its demand location or scale. The same product may have different behavior in different contexts. For instance, retailers that have their main competitive advantage in their geographical location usually see a change in the scale of the demand of their products (Agrawal and Seshadri, 2000). The multiplicative demand curve is particularly convenient because it assumes that the price elasticity of the demand remains constant at every price, unlike the linear demand function, that presents much more negative elasticities at very low prices (Li et al., 2014). Moreover, when converted to a logarithmic scale it allows an easy estimation of the parameters via regression techniques (Shi and Guo, 2012, Monahan et al., 2004). The isoelastic demand is widely used for measuring supply and demand in agricultural products: AGRISIM (Agricultural Simulations Model) uses demand and supply models with constant elasticities to model multi-region multi-commodity flow of agricultural goods. This model was used for assessing European Union's agricultural policies and bilateral trade liberalization between EU-member states and non-member states in the Mediterranean basin within the frame of the MEDFROL project (Kavallari et al., 2006, Britz and Heckelei, 2008), that includes the analysis of commodities such as rice, olive oil, wheat, and tomatoes among others. However, when it comes to using this demand model in optimization, one of its main drawbacks is its lack of tractability, especially when compared to the much simpler linear demand model. We will see how its application to our problem is no exception.

In our newsvendor problem, the price and the order quantity are usually selected in order to maximize the expected return. As commented above, such an approach is typical in the literature (Karlin and Carr, 1962, Mills, 1959, Petruzzi and Dada, 1999, Xu et al., 2011, Kocabıyıkoğlu and Popescu, 2011, Xu et al., 2010, Wang et al., 2004, Rubio-Herrero et al., 2015, Rubio-Herrero and Baykal-Gürsoy, 2018). We perform this optimization in the light of risk considerations. This has also been studied before for joint price and stock optimization under mean-variance and lower semi-deviation (Choi and Chiu, 2012), under CVaR (Chen et al., 2009), and under a utility framework (Agrawal and Seshadri, 2000), although not for risk-seeking individuals. In this paper, we present a model based on Markowitz's mean-variance tradeoff (Markowitz, 1952). Such a model penalizes the expected profit with the variance of the profit according to a risk factor λ. The sign of this parameter reveals the attitude of the newsvendor towards the variance of the profit: a positive parameter penalizes volatility in the profit (risk-averse); a negative parameter favors volatility in the profit (risk-seeking). The former is commonly found in the literature; the latter is in general much scarcer and we do not know of many papers that have studied this problem in risk-seeking situations, except for some earlier attempts on simpler newsvendor models with one decision variable (Choi et al., 2008) and some recent publications (Rubio-Herrero et al., 2015, Rubio-Herrero and Baykal-Gürsoy, 2018, Nagarajan and Shechter, 2013, Long and Nasiry, 2014, Xinsheng et al., 2015). Despite the surge of downside risk measures in the last few years, it is not clear that they have theoretical advantages over the mean-variance tradeoff (Grootveld and Hallerbach, 1999). Also, the use of mean-variance models remains the standard in industry because of its conceptual simplicity, clarity, and computational advantages over these downside risk measures.

The analysis of this problem under risk-seeking behavior is justified by Prospect Theory (Kahneman and Tversky, 1979): humans are usually *loss-averse* (not risk-averse) and make decisions in term of losses. Risk-seeking attitudes may arise in situations in which individuals that have lost an important amount of money want to recoup their losses in one lucky strike. In other words, individuals will show a risk-seeking behavior if they are in a state of loss or if the riskier option offers the possibility of eliminating loss (Scholer et al., 2010). A state of loss is defined with respect to a target return, notion that was introduced by Fishburn (1977). It is typical that people that see returns below this target develop risk-seeking behaviors, whereas people that see returns over it develop risk-averse behaviors. In terms of utility functions, this can be seen as convex and concave utility functions respectively, depending on which side of the target level we are on. On a corporate level, Bowman (1982) showed that this individual behavior is transferable to industry and that companies with financial troubles take larger risks. A very interesting summary of behavioral studies that suggest a negative relationship between performance and risk taking is presented in Fiegenbaum and Thomas (1988).

The mean-variance measure of risk does not fall within the category of *coherent measures* as defined in Artzner et al. (1999): in particular the variance does not possess any of the four characteristics that characterize such risk measures, namely, subadditivity, monotonicity, translation equivariance, and positive homogeneity. Nevertheless, similarly to another commonly used measure like VaR (also not subadditive and therefore not coherent (Szegö, 2005)), mean-variance models have been widely used in the literature for incorporating risk to decision-making (Chen and Federgruen, 2000, Choi et al., 2008).

It is worth remarking that the mean-variance price-setting newsvendor model with isoelastic, price-dependent demand was studied by Choi and Chiu (2012); nevertheless, in this work based in the fashion industry, the authors assume that the pricing and stocking decisions are separable in time. The retailer first decides the order quantity before the selling season and then decides the optimal selling price during the selling season, once the stochastic portion of the price-dependent demand is known. Instead, we propose the joint optimization approach that Rubio-Herrero et al. (2015) and Rubio-Herrero and Baykal-Gürsoy (2018) followed when assessing respectively the concavity and the unimodality of the objective function for a risk-sensitive profit maximizer when the demand is additive. This approach will be applied now to the case of multiplicative demand. The goals pursued in our work are to: extend the framework that Kocabıyıkoğlu and Popescu (2011) presented for price-inelastic products to price-elastic products by considering the unimodality of the objective function in the price-setting newsvendor problem with isoelastic, price-dependent demand; find conditions for such unimodality as a function of the risk sensitivity of the decision maker: risk-neutral, risk-averse, and risk-seeking individuals; present these conditions under mild assumptions, so the generality of the problem presented can hold above other considerations; finally, write these conditions in managerial terms to give them a more practical meaning. This poses an important difference with respect to previous works under the multiplicative demand model, in which conditions are given in much more technical terms involving failure rates and generalized failure rates. Our results will be given in terms of the lost sales rate (LSR) elasticity, a concept which directly relates to the level of service given to the customer, even though it depends implicitly on the failure rate of the random term of the demand.

The paper is organized as follows: In §2 we introduce the problem formulation and the methodology that will be used; in §3 and §4 we apply this methodology to the optimization of the price and the stock factor respectively; in §5 we present a sensitivity analysis of the objective function; finally we summarize our findings in §6.

2. Problem Formulation

Consider the single-stage, single-product, newsvendor problem with two decision variables, namely stock quantity and price. Risk-neutral retailers would thus pursue the maximization of their expected profit, $\mathbb{E}(\Pi(p,x)) = p\mathbb{E}(min\{D(p,\epsilon),x\}) - cx$, where $c \in \mathbb{R}^+$ is the cost of the product and $D(p,\epsilon)$ is its demand. The first term represents the income collected in a single-period, whereas the second term represents the cost incurred when manufacturing or procuring x units of product. We assume that such a cost increases linearly with the number of units procured or manufactured. The demand is multiplicative, price-dependent, and stochastic, i.e. $D(p,\epsilon) = y(p)\epsilon$, where ϵ is a random variable characterized by its probability density function $f(\cdot)$ and its cumulative density function $F(\cdot)$.

The term $y(p)$, is a decreasing function of the price and represents the deterministic portion of the demand or *riskless demand* (Petruzzi and Dada, 1999). In the case of an isoelastic demand, i.e., $y(p) = ap^{-b}$ with $a \in \mathbb{R}^+$ and $b > 1$. Since we use the term *elasticity* for referring to the price elasticity of demand, $b > 1$ represents products that are commonly known to present a constant, elastic demand, meaning that an increase of 1% in the price of the commodity in question will always produce a drop in its demand that is equal to b, greater than 1%. Examples of elastic goods are products that are not critically needed by consumers or for which they can readily find a substitute. The archetypal example is the elasticity of soft drinks like Coca-Cola or Mountain Dew, that have elasticities of 3.8 and 4.4, respectively (Ayers and Collinge, 2003). On the contrary, goods that see a reduction of their demand by less than 1% after an increase of their price are called inelastic (i.e. $b < 1$). Examples of such goods are alcoholic beverages or cigarettes. These types of goods will not be covered in this paper.

The objective function is presented as a combination of the expected profit and the variance of the profit weighted with a risk parameter. The sign of this parameter determines whether the newsvendor is risk-averse or risk-seeking. If $\lambda > 0$ (i.e., the variance of the profit decreases the objective function), the newsvendor is risk-averse; if $\lambda < 0$ (i.e., the variance of the profit increases the objective function), the newsvendor is risk-seeking.

$$\tilde{P}(p, x) = \quad \mathbb{E}(\Pi(p, x)) - \lambda Var(\Pi(p, x)) \quad = p\mathbb{E}(min\{D(p, \epsilon), x\}) - cx - \lambda Var(p \cdot min\{D(p, \epsilon), x\}).$$

This problem is, of course, the unconstrained parametrized version of the problem

$$\begin{aligned} \max_{p,x} \quad & p\mathbb{E}\left[min\{D(p, \epsilon), x\}\right] - cx \\ \text{s.t.} \quad & Var\left[(p \cdot min\{D(p, \epsilon), x\}\right] \lesseqgtr \gamma, \end{aligned}$$

after defining the corresponding Lagrangian. The constrained problem is solved for a particular value of γ. The unconstrained problem, parametric programming formulation, in turn, fixes the value λ^* of the Lagrange multiplier and finds the duple (x^*, p^*) for which (x^*, p^*, λ^*) is the optimal solution of the constrained problem for some γ. It is well known that if (x^*, p^*) is a solution for the unconstrained problem, then it is a solution for the constrained problem with $\gamma = Var(\Pi(x^*, p^*))$, and at the optimal solution, $\lambda = \frac{\partial E(\Pi(x,p))}{\partial Var(\Pi(x,p))}$ (Li and Ng (2000)). Moreover, this formulation may be preferable by the retailers since they are able to specify their desirable tradeoff between the expected profit and the associated risk.

A common feature of the classic newsvendor model is the use of a salvage value at which the excess of stock can be sold at the end of the period, This salvage value, s, is such that $p > c > s \geq 0$ and its effect can be included in the model presented above by simply redefining a new cost $\bar{c} = c - s > 1$ and a new price $\bar{p} = p - s$ (Choi and Ruszczyński, 2008).

Throughout this paper, we make several general assumptions on some parameters and functions, which we outline below. It is our aim that these assumptions interfere as little as possible with obtaining a framework for the risk-sensitive mean-variance newsvendor problem that is as applicable as possible.

Assumption 1: ϵ is a random variable with finite variance $Var(\epsilon)$ and compact and convex support $[A, B]$, $0 < A < 1 < B$.

Assumption 2: $F(\cdot)$ is twice differentiable with continuous second derivative.

Assumption 3: $\mathbb{E}(\epsilon) = 1$, thus $E[D(p, \epsilon)] = y(p)$.

Assumption 4: $c \geq 1$.

We consider assumptions 1 and 2 to be mild: if ϵ is a random variable defined in an open interval, like an exponential distribution or a normal distribution, a suitable truncation can be performed. On the other hand, most of the continuous random variables that are used in inventory problems are twice differentiable with continuous second derivative. Finally, assumptions 3 and 4 hold WLOG since the parameter a can absorb any value of $\mathbb{E}(\epsilon)$ different from 1 if needed and since any currency can be easily reconverted to a new scale such that $c \geq 1$.

Following Petruzzi and Dada (1999), we define the price-sensitive *stock factor* $z = x/y(p)$. Since $x \in [y(p)A, y(p)B]$ (i.e. the order quantity must be within the range defined by the possible demand values), it follows that $z \in [A, B]$. We can rewrite the objective function as a function of (p, z):

$$\begin{aligned}\tilde{P}(p, x) &= p\mathbb{E}(min\{D(p, \epsilon), x\}) - cx - \lambda Var(p \cdot min\{D(p, \epsilon), x\}) \\ &= y(p)p\mathbb{E}(min\{\epsilon, z\}) - czy(p) - \lambda Var(py(p) \cdot min\{\epsilon, z\}) \\ &= py(p)\mu(z) - czy(p) - \lambda(y(p)p)^2\sigma^2(z) \\ &= \mathbb{E}(\Pi(p, z)) - \lambda Var(\Pi(p, z)) =: P(p, z), \qquad (1)\end{aligned}$$

where $\mu(z) = \mathbb{E}(min\{\epsilon, z\}) = \int_z^B (z-u) f(u) du + 1$ and, $\sigma^2(z) = Var(min\{\epsilon, z\}) = Var(\epsilon) + \int_z^B (z^2 - u^2) f(u) du - \mu^2(z) + 1,\ z \in [A, B]$. The derivation of these two expressions can be found in the appendix. Note that $\mu(\cdot)$ is always an increasing, concave function, for $\mu'(z) = 1 - F(z)$ and $\mu''(z) = -f(z)$, $z \in [A, B]$. On the other hand, $\sigma^2(\cdot)$ is also an increasing function with $\sigma^{2'}(z) = 2\,(1 - F(z))\,(z - \mu(z))$, although not much can be said about the sign of its second derivative $\sigma^{2''}(z) = 2(1 - F(z))F(z) - 2f(z)(z - \mu(z))$.

Since they will be used frequently throughout this paper, we next present some partial derivatives of the objective function $P(p,z) = ap^{-b}\left(\mu(z)p - cz - \lambda a p^{2-b}\sigma^2(z)\right)$ with respect to (p,z):

$$\begin{aligned}
\frac{\partial P(p,z)}{\partial p} &= ap^{-(b+1)}\left(2\lambda\sigma^2(z)a(b-1)p^{-(b-2)} - (b-1)\mu(z)p + bcz\right), && (2)\\
\frac{\partial^2 P(p,z)}{\partial p^2} &= ap^{-(b+2)}(-2(b-1)\lambda\sigma^2(z)a(2b-1)p^{-(b-2)} + b(b-1)\mu(z)p - (b+1)bcz), && (3)\\
\frac{\partial^2 P(p,z)}{\partial p \partial z} &= ap^{-(2b-1)}\left(bcp^{b-2} - (b-1)p^{b-1}\mu'(z) + 2(b-1)a\lambda\sigma^{2'}(z)\right). && (4)
\end{aligned}$$

The optimization procedure that we will follow in this paper is typical in the literature when tackling the joint optimization of the newsvendor problem (Zabel, 1970): the objective function $P(\cdot,\cdot)$ will be optimized by first obtaining the optimal price for a given stock factor, $p^*(z)$; then, this objective function will be rewritten as a function of only one variable: $P^*(z) := P(p^*(z),z) = ap^*(z)^{-b}(\mu(z)p^*(z) - cz - \lambda a p^*(z)^{2-b}\sigma^2(z))$; finally, $P^*(\cdot)$ will be maximized to find the optimal pair $(z^*, p^*(z^*))$.

3. Optimization with Respect to p

Given a stock factor z, finding the function $p^*(z)$ reduces to dentifying the critical points of $P(\cdot,\cdot)$ by solving the condition $\partial P/\partial p = 0$, i.e.

$$2\lambda\sigma^2(z)a(b-1)p^*(z)^{-(b-2)} - (b-1)\mu(z)p^*(z) + bcz = 0. \qquad (5)$$

Clearly, for any given λ, finding a closed-form for $p^*(z)$ requires knowledge of b as well. An exception to this rule is the risk-neutral case ($\lambda = 0$), for which the optimal price is always $p^*(z) = bcz/((b-1)\mu(z))$. We refer the reader to Petruzzi and Dada (1999) for a thorough analysis of this case. This differs from the additive demand case, in which the optimal price is always defined in closed form, and makes an enormous difference when it comes to finding constant bounds for the conditions for the unimodality of $P^*(\cdot)$, as the next step in the sequential optimization process is to substitute $p^*(z)$ in the objective function. However, when the demand is isoelastic and the newsvendor is risk-sensitive, we may find multiple solutions to equation (5). Our goal is to be able to identify a positive real solution, greater than c, that maximizes the performance measure $P(\cdot,z)$ for a given stock factor z. In the subsections that follow we study how to identify such a solution, but in the meantime we include in Table 1 below some closed-form solutions of the optimal price for specific values of the demand elasticity. Note that for $b = 2$ and $b = 3$ there are some values of λ for which $p^*(z)$ is a negative real number or an imaginary number. We will see later on in which circumstances and for which values of the risk parameter this occurs. The proofs of all the results that follow are provided in the appendix.

b	$p^*(z)$
1.5	$\left(\dfrac{a\lambda\sigma^2(z) + \sqrt{(a\lambda\sigma^2(z))^2 + 3\mu(z)cz}}{\mu(z)}\right)^2$
2	$\dfrac{2cz + 2\lambda a\sigma^2(z)}{\mu(z)}$
3	$\dfrac{3cz + \sqrt{(3cz)^2 + 32\lambda a\sigma^2(z)\mu(z)}}{4\mu(z)}.$

Table 1: Closed-form solutions of some optimal prices

3.1. Risk-averse retailer

When the retailer is risk-averse ($\lambda > 0$) equation (5) will only have one real positive solution, and such a solution will always be greater than the cost c and will also represent a maximizer of $P(\cdot, z)$. This is a remarkable result because despite not having a closed form for the optimal price, it allows us to have certainty about the behavior of the performance measure $P^*(\cdot)$ that we will use later on.

Theorem 1. *If the retailer is risk-averse ($\lambda > 0$), then, for a given stock factor z there is exactly one positive real solution to equation (5), $p^*(z)$. Moreover, the objective function $P(\cdot, z)$ is unimodal with respect to p in $(0, \infty)$ $\forall z \in [A, B]$, and the critical point $p^*(z)$ is a maximizer. Furthermore, the optimal price $p^*(z)$ is always greater than c.*

Note that, since $\sigma^2(A) = 0$, the optimal price at $z = A$ is independent of the level of risk aversion and equals $cb/(b-1)$. This price depends only on the elasticity of the demand of the product and its cost, thus matching the results by Wang et al. (2004), Petruzzi and Dada (1999).

3.2. Risk-seeking retailer

When the newsvendor is risk-seeking ($\lambda < 0$) the number of positive real roots depend on the value of b. The results yielded for this case are more complicated, but we can still predict the number of positive real roots, whether they maximize the performance measure $P(\cdot, z)$ for a given stock factor z, and whether they are contained in the range $[c, \infty)$. For the study of risk-seeking instances, we introduce one extra assumption.

Extra assumption for the risk-seeking case: The study of risk-seeking cases is restricted to *moderately risk-seeking instances*. An instance is considered moderately risk-seeking if $\lambda \in [\lambda_{lim}, 0)$. We define λ_{lim} as

$$\lambda_{lim} = \max_{z\in[A,B]} t(z), \tag{6}$$

where we introduce the function $t : [A, B] \to \mathbb{R}^-$ as

$$t(z) \quad = \quad \frac{(b-1)\mu(z) - bz}{2a(b-1)\sigma^2(z)} c^{b-1}.$$

The value of the function $t(\cdot)$ represents the value that, given a stock factor z, the risk parameter λ has when the critical point $p^*(z)$ is equal to c. We propose the following statement:

Theorem 2. *Let $\lambda < 0$. Then, for each stock factor z:*

(i) If $1 < b < 2$, or $b = 2$ and $\lambda \geq \lambda_{min}$, with $\lambda_{min} = \max_{z\in[A,B]} -cz/\left(a\sigma^2(z)\right)$, then there is exactly one positive real solution $p^*(z)$ to equation (5). Moreover, the objective function $P(\cdot, z)$ is unimodal with respect to p in $(0, \infty)$ and $p^*(z)$ is a maximizer.

(ii) If $b > 2$, there are either two positive real solutions to equation (5), or there are none. The objective function $P(\cdot, z)$ is bimodal with respect to p in $(0, \infty)$ if $b > 2$ and equation (5) has two positive real solutions. In this case, let those roots be $p_1(z)$ and $p_2(z)$ such that $0 < p_1(z) < p_2(z)$. Then, $p_1(z)$ is the minimizer of the profit function, $P(\cdot, z)$, for constant z, while $p_2(z)$ is the maximizer. Thus, $p^*(z)$ refers to $p_2(z)$.

Theorem 2 guarantees that the roots of (5) are contained in $(0, \infty)$. However the threshold value λ_{lim} represents the values of λ above which c is not a solution to (5) for any stock factor z in [A,B], as stated in the next lemma.

Lemma 1. *If there is a positive root $p^*(z)$ that maximizes $P(\cdot, z)$, and $\lambda \in (\lambda_{lim}, 0)$, then $p^*(z)$ is greater than $c, \forall z \in [A, B]$.*

In view of the results from Theorem 2 and Lemma 1, we can summarize the different possibilities in the optimization of $P(\cdot, z)$ in $[c, \infty)$ for a given stock factor in moderately risk-seeking instances. This is shown in Table 2.

3.3. Examples

Let $D(p, \epsilon) = 10^6 p^{-b}\epsilon$ with $\epsilon \sim U[0.001, 1.999]$. Let $c = 100$. The first-order condition (5) as a function of b and z is $2\cdot 10^6\lambda(b-1)\sigma^2(z)p^{-(b-2)} - (b-1)\mu(z)p + 100bz = 0$, with $\mu(z) = -0.2503z^2 + 1.0005z - 2.503\cdot 10^{-7}$ and $\sigma^2(z) = -0.06263z^4 + 0.1671z^3 - 5.009\cdot 10^{-4}z^2 + 5.009\cdot 10^{-7}z - 2.001\cdot 10^{-10}$.

- <u>$b = 1.5$:</u> the first-order condition becomes $10^6\lambda\sigma^2(z)\sqrt{p} - \frac{1}{2}\mu(z)p + 150z = 0$. The lower bound for λ, as shown in Lemma 1, can be obtained numerically. It turns out that (6) attains its maximum at $z = 1.7215$ with a value of $\lambda_{lim} = \max(\mu(z) - 3z)5\cdot 10^{-6}/\sigma^2(z) = -6.952\cdot 10^{-5}$. The only positive real root in this case is given by $p^*(z) = \left(10^6\lambda\sigma^2(z) + \sqrt{(10^6\lambda\sigma^2(z))^2 + 300\mu(z)z}\right)^2/\mu(z)^2$.

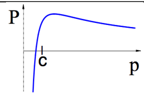
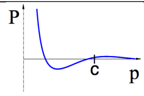
Case	b	Roots in $\mathbb{R}^+$	Max. in $[c, \infty)$	Shape of $P(\cdot, z)$
1	$1 < b \leq 2$	1	$p^*(z)$	
2	$b > 2$	2	$p^*(z)$	

Table 2: Analysis of the optimality in $[c, \infty)$ of the objective function with respect to the price for a given stock factor in moderately risk-seeking cases

- $\underline{b = 2}$: the first-order condition becomes $2 \cdot 10^6 \lambda \sigma^2(z) - \mu(z) p + 200z = 0$. Since $b = 2$, per the corollary from Lemma 1 we use (6) to set a lower bound for λ and thus set $\lambda_{lim} = \max(\mu(z) - 2z) 5 \cdot 10^{-5} / \sigma^2(z) = -4.085 \cdot 10^{-4}$. The only positive real root is $p^*(z) = 2\left(10^6 \lambda \sigma^2(z) + 100z\right) / \mu(z)$.

- $\underline{b = 3}$: the first-order condition becomes $4 \cdot 10^6 \lambda \sigma^2(z)/p - 2\mu(z) p + 300z = 0$, $\lambda_{lim} = -0.02639$ and the equation above has two roots: $p_1^*(z) = \left(300z - \sqrt{9 \cdot 10^4 z^2 + 32 \cdot 10^6 \mu(z) \sigma^2(z) \lambda}\right) / (4\mu(z))$, and $p_2^*(z) = \left(300z + \sqrt{9 \cdot 10^4 z^2 + 32 \cdot 10^6 \mu(z) \sigma^2(z) \lambda}\right) / (4\mu(z))$.

  As mentioned before, when $\lambda = \lambda_{lim}$, c is a root of (5). Further analysis shows that this root occurs at $z = 1.6214$. However, in this case this point corresponds to a minimizer (i.e. $p_1^*(z) = c$). For this reason, the right-most graph in Figure 1 does not show a curve that reaches $p^*(z) = p_2^*(z) = c = 100$ when $\lambda = \lambda_{lim}$. This can be seen in further detail in Figure 2, where $c = 100$ is only a root of (5) when $\lambda = \lambda_{lim}$ (in this case at $z = 1.6214$), but this root does not correspond to a maximizer.

4. Optimization with Respect to z

As commented in the introduction, it is usual in the literature to find examples based on different risk measures that guarantee the existence of a unique maximum of the objective function under more restrictions, usually related to the generalized failure rate of ϵ. For instance, Xu et al. (2011), Wang et al. (2004) show unimodality for the risk-neutral case and multiplicative demand models if the random variable has an increasing generalized failure rate. For risk-averse cases with CVaR considerations, Chen et al. (2009) show that a strictly increasing generalized failure rate in the risk distribution is necessary to attain unimodality. In what follows, we proceed to introduce different conditions for unimodality depending on the risk-parameter

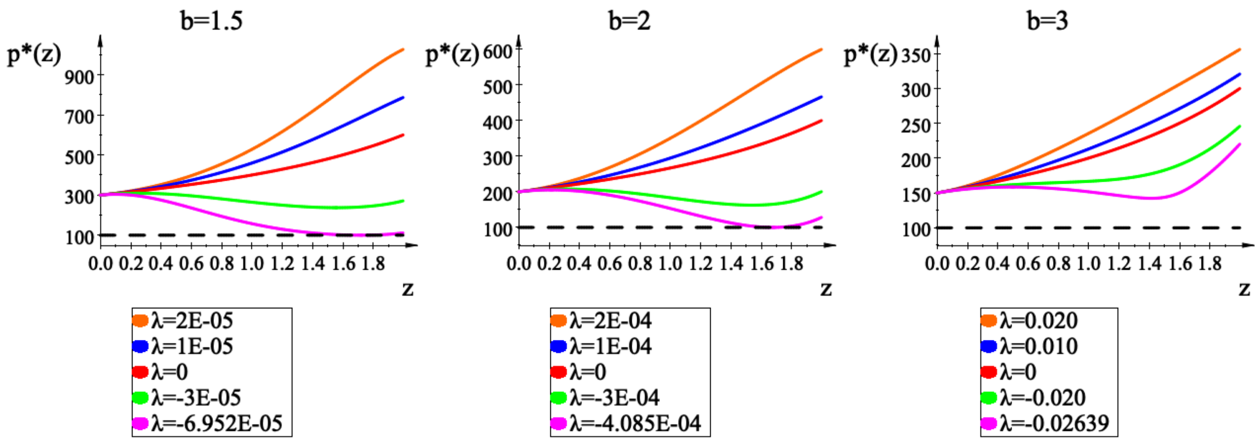


Figure 1: Optimal price function under different risk scenarios

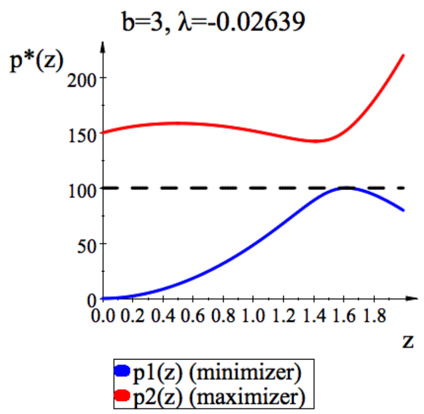


Figure 2: Maximizing and minimizing prices for $\lambda = \lambda_{lim}$

in a mean-variace setting. We want to give a more practical, managerial meaning to our results and, to that end, we will define the lost sales rate (LSR) elasticity:

Definition 1. *(Kocabıyıkoğlu and Popescu, 2011) The lost sales rate (LSR) elasticity for a given price p and inventory level x is defined as*

$$\tilde{\kappa}(p, x) \quad = \quad \frac{p(G(p, x))'_p}{1 - G(p, x)},$$

where $G(p, x) := \mathbb{P}(D(p, \epsilon) \leq x)$ and $(G(p, x))'_p \equiv \partial G(p, x)/\partial p$.

In particular, when the demand is multiplicative, $\mathbb{P}(y(p)\epsilon \leq x) = \mathbb{P}(\epsilon \leq x/y(p)) =: F(z)$, and therefore we obtain that

$$\tilde{\kappa}(p, x) = \frac{p(G(p, x))'_p}{1 - G(p, x)} = \frac{bzf(z)}{1 - F(z)} \quad =: \tilde{\varepsilon}(z). \tag{7}$$

Two major insights can be derived from here: on the one hand, by definition, the level of service is to the LSR elasticity what the demand is to the price elasticity. That is, given that $\mathbb{P}(D(p,\epsilon) \leq x)$ shows the probability of servicing the demand, the LSR elasticity tells us what is the change in this probability when we increase the price of our product, for a given stock factor. This is analogous to the price elasticity of demand, which states the change in the demand when there is an increase in the price. On the other hand, and just like the price elasticity of demand in isoelastic demand curves, the LSR elasticity is not a function of the price when the demand is multiplicative. In other words: the price-isoelastic demand is also *LSR-isoelastic* because, given a stock factor, the change in the level of service will be the same regardless of the price from which that increase takes place.

In order to continue with the optimization process, we define the objective function $P^*(\cdot)$ as a function of z and derive its first-order derivative as follows:

$$P^*(z) := \quad P(p^*(z), z) \quad = ap^*(z)^{-b}\left(\mu(z)p^*(z) - cz - \lambda a p^*(z)^{2-b}\sigma^2(z)\right), \tag{8}$$

$$P^{*'}(z) \quad = \quad ap^*(z)^{-b}R(z), \tag{9}$$

where $R(z) = (1 - F(z))p^*(z) - c - \lambda a p^*(z)^{-(b-2)}\sigma^{2'}(z).$ (10)

When calculating (9), it was taken into account that $p^*(z)$ satisfies (5), and therefore $\partial P(p^*(z), z)/\partial p = 0$.

In general, the function $P^*(\cdot)$ cannot be fully characterized. For a given value of λ, the optimal price does not have a closed-form solution that is valid for all values of b. Instead, for each price-elasticity of the demand, the optimal price will present a different formula, as shown on Table 1. An exception to this is the risk-neutral ($\lambda = 0$) optimal price $p^*(z) = bcz/((b-1)\mu(z))$. This fact introduces additional complexity when analyzing the behavior of $P^*(\cdot)$ and makes it more complicated to attain constant bounds in terms of the LSR elasticity, which is our ultimate goal. In the next subsections, we perform a first approach to obtaining these bounds. For the risk-neutral retailer, the complete characterization of $P^*(\cdot)$ allows us to obtain those constant bounds. For risk-sensitive retailers, we will obtain three different bounds, with only one of them being constant.

4.1. Risk-neutral retailer

When $\lambda = 0$, $P^*(\cdot)$ and its first-order derivative can be greatly simplified to $P^*(z) = ap^*(z)^{-b}(\mu(z)p^*(z) - cz)$ and $P^{*'}(z) = ap^*(z)^{-b}R(z)$, with $R(z) = (1 - F(z))p^*(z) - c$. This is the same result obtained by Wang et al.

(2004) and Petruzzi and Dada (1999). It is thus clear that the optimal stock factors z^* of the risk-neutral, single-stage newsvendor problem with isoelastic demand satisfy the equation $F(z^*) = 1 - c/p^*(z^*)$. When the stock factor is the only decision variable, this result particularizes for the well-known result of the single-stage newsvendor problem where the stock factor that maximizes the profit is unique and equal to the $(1 - c/p)$-quantile of z. However, when the price is also a decision variable it is not clear anymore whether this equation has one or multiple solutions. The following theorems intend to shed some light on the conditions that guarantee local and global optimality of the solutions to $R(z) = 0$:

Theorem 3. *The following local and global optimality conditions hold for the risk-neutral case:*

a) (Local optimality- sequential decision process) Let z^* be a solution to the equation $F(z) = 1 - c/p^*(z)$. If $\tilde{\varepsilon}(z^*) > 1$, then the pair $(p^*(z^*), z^*)$ is a strict local maximum of $P(\cdot,\cdot)$ in $[c,\infty) \times [A, B]$. If $\tilde{\varepsilon}(z^*) < 1$, this pair is a saddle point of $P(\cdot,\cdot)$ in $[c,\infty) \times [A, B]$.

b) (Global optimality - sequential decision process) If $\tilde{\varepsilon}(z) > 1$, $\forall z \in [A, B]$, then $P^*(\cdot) \equiv P(p^*(\cdot),\cdot)$ is unimodal and there is a unique stock factor z^* which maximizes $P^*(z)$: therefore the pair $(p^*(z^*), z^*)$ solves the risk-neutral, single-stage newsvendor problem with isoelastic demand.

c) (Global optimality - simultaneous optimization) If $\tilde{\varepsilon}(z) > 1$, $\forall z \in [A, B]$, then there exists a unique pair (p^{**}, z^{**}) that maximizes $P(p, z)$, hence solves the risk-neutral, single-stage newsvendor problem with isoelastic demand.

Example: Consider the demand function $D(p,\epsilon) = 10^6 p^{-3}\epsilon$. Let $c = 50$. The random variable ϵ has a probability density function denoted by $f(z) = 0.5f_1(z) + 0.5f_2(z)$, where $f_1(\cdot)$ and $f_2(\cdot)$ are in turn the probability density functions of two normal random variables with means $0.4, 1.6$ and standard deviations $0.1, 0.2$, respectively. We assume $A = 0.001$ and $B = 3$ because the density of ϵ beyond those points is negligible. The optimal price for each value of z can be calculated by using the third entry of Table 1. Solving the equation $F(z) = 1 - c/p^*(z)$ numerically yields the following solutions: $z_1^* = 0.4831, z_2^* = 0.8, z_3^* = 1.392$. Evaluating these points in the expression $\tilde{\varepsilon}(z) = 3zf(z)/(1 - F(z))$ yields the following LSR elasticities: $\tilde{\varepsilon}(z_1^*) = 3.4026$, $\tilde{\varepsilon}(z_2^*) = 0.0048$, $\tilde{\varepsilon}(z_3^*) = 5.6998$.

These results show that the points $(p^*(z_1^*), z_1^*) = (83.1294, 0.4831)$ and $(p^*(z_3^*), z_3^*) = (117.5295, 1.392)$ are strict local maxima of $P(\cdot,\cdot)$, whereas $(p^*(z_2^*), z_2^*) = (100, 0.8)$ is a saddle point of $P(\cdot,\cdot)$. The pair $(117.5295, 1.392)$ is also the global maximum of $P(\cdot,\cdot)$ in $[50,\infty) \times [0.001, 3]$ with a value of the objective function $P(p^*(z_3^*), z_3^*) = 21.4355$. Figure 3 shows these three points as the solutions to $P^{*'}(z) = 0$ (i.e. as the solutions to $F(z) = 1 - c/p^*(z))$ plotted on the curve $P^*(z) = P(p^*(z), z)$ and then those three points on the surface defined by $P(p, z)$. Note that (0.8, 100) is a local minimum of $P^*(\cdot)$ but it is a saddle point of $P(\cdot, \cdot)$.

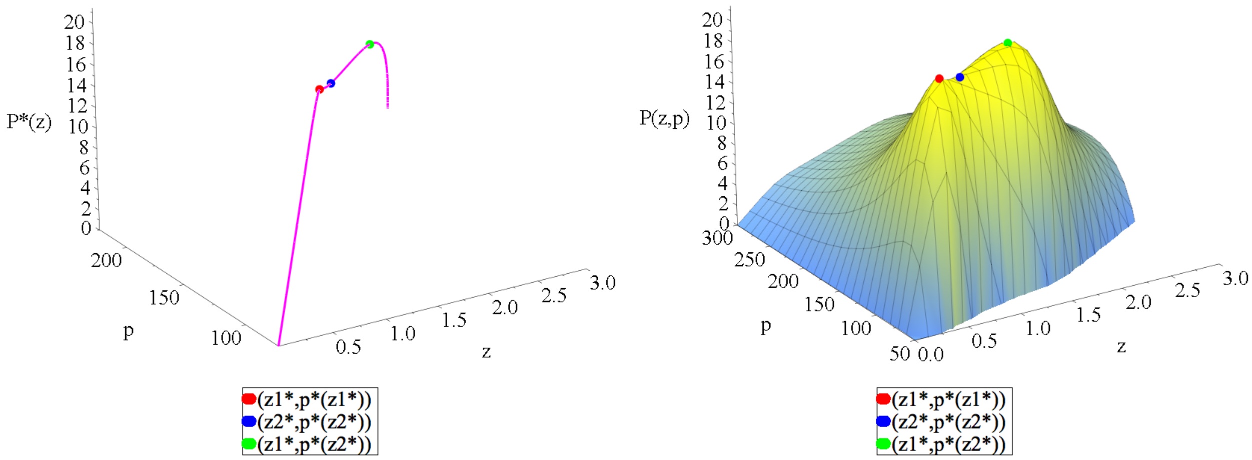


Figure 3: Illustration of local optimality conditions for the risk-neutral case

The theorem above gives conditions for a point to be either a local maximum or a unique maximum of $P(\cdot, \cdot)$ in the risk-neutral, single-stage newsvendor problem with multiplicative demand. Some similar results that guaranteed the existence of a unique maximum in this problem were obtained in the past as a function of the *failure rate* of ϵ, $h(z) = f(z)/\left(1 - F(z)\right)$, and the *generalized failure rate* of ϵ, $g(z) = zh(z)$.

Remark 1. *Petruzzi and Dada (1999) show that if $b \geq 2$ and $2h(z)^2 + h'(z) > 0$ this problem has a unique solution. Wang et al. (2004) claim that ϵ having an increasing generalized failure rate is sufficient, thus uncoupling the economic parameters of the model from the uniqueness of the optimal solution. Both conditions are the consequence of imposing the unimodality of equivalent formulations of $R(z)$ (see both papers for further details). In contrast, we make the Hessian of $P(\cdot, \cdot)$ negative definite in all the pairs $(z^*, p^*(z))$ for proving our local optimality condition and transform the results to give them the more economic and managerial interpretation that the LSR elasticity provides. Note, however, that LSR elasticity is proportional to the generalized failure rate in the case of multiplicative demand, for $\tilde{\varepsilon}(z) = bg(z)$. Therefore, the above condition corresponds to $g(z) > \frac{1}{b}$, in terms of the generalized failure rate.*

Remark 2. *This result also complements Theorem 2 in Kocabıyıkoğlu and Popescu (2011) that claims the concavity of the objective function in risk-neutral cases if $\tilde{\varepsilon}(x) > 1/2$. However, they assume that $2y'(p) + py''(p) < 0$, which implies that the good has an inelastic demand ($b < 1$). In this paper, we extend* *the concept of concavity to that of unimodality and we consider products that have an elastic demand ($b > 1$).*

4.2. Risk-sensitive retailer

When $\lambda \in [\lambda_{lim}, \infty)$, $P^*(\cdot)$ and its first-order derivative can be written as shown in equations (8) - (10). There exist some conditions under which the unimodality of the risk-sensitive problem, either risk-averse or moderately risk-seeking, is guaranteed. We first proceed with two definitions introduced in Rubio-Herrero and Baykal-Gürsoy (2018) and then with the theorem that summarizes our findings.

Definition 2. *The elasticity of the optimal price is defined as:*

$$\rho(z) \quad := \quad \frac{dp^*(z)}{dz}\frac{z}{p^*(z)}. \tag{11}$$

A priori, it is not possible to know the sign of $\rho(z)$, as that requires knowledge of $p^*(\cdot)$ and its derivative.

Definition 3. *The elasticity of the expected stock factor surplus is defined as:*

$$\omega(z) \quad := \quad \frac{d\mathbb{E}\left[(z-\epsilon)^+\right]}{dz}\frac{z}{\mathbb{E}\left[(z-\epsilon)^+\right]}. \tag{12}$$

In the case of multiplicative demand, as in the case of additive demand, $\mathbb{E}\left[(z-\epsilon)^+\right] \equiv z - \mu(z)$ and therefore $\omega(z) = \frac{zF(z)}{z-\mu(z)} \geq 0, \forall z \in [A, B]$.

Theorem 4. *(Global optimality) Let $\eta(z) = (1-F(z))p^*(z) - c$. The following sufficient conditions guarantee the unimodality of the single-stage newsvendor problem with isoelastic demand (i.e. there is only one stock factor z^* that satisfies the equation $R(z) = 0$ and the pair $(z^*, p^*(z^*))$ maximizes $P(\cdot,\cdot)$):*

a) If $\lambda \in [\lambda_{lim}, \infty)$ (risk-sensitive newsvendor):

$$\tilde{\varepsilon}(z) \quad > \quad b\rho(z) + \frac{b}{c}\eta(z)\left[(b-1)\rho(z) - \omega(z)\right], \ \forall z \in [A, B],$$

b) If $\lambda \in [0, \infty)$ (risk-averse and risk-neutral newsvendor):

$$\tilde{\varepsilon}(z) \quad > \quad \left(1 + \frac{(b-1)\eta(z)}{c}\right)^2, \ \forall z \in [A, B], \tag{13}$$

c) If $\lambda \in [0, \infty)$ (risk-averse and risk-neutral newsvendor) and $b \geq 2$:

$$\tilde{\varepsilon}(z) \quad > \quad \left(1 + \frac{2\lambda a(b-1)(B-1)}{c^{b-1}}\right)^2, \ \forall z \in [A, B], \tag{14}$$

A very interesting remark to make here is that the global optimality condition of Theorem 3 is a particularization of the global optimality condition of Theorem 4. As a matter of fact, when $\lambda = 0$, it turns out that $\eta(z)|_{R(z)=0} = 0$ (because in this case $\eta(z) = R(z)$). Applying this to (13) yields directly the expression $\tilde{\varepsilon}(z) > 1$. On the other hand, the bound provided by (14) is in general very close to 1. This is because the order of magnitude of λ is generally very small: Let the order of magnitude of the objective function $P(\cdot,\cdot)$ be $\sim 10^m$ (typically measured in some currency); the order of magnitude of the variance of the profit is thus $\sim 10^{2m}$, which results in λ being in the order of $\sim 10^{-m}$; the values of the price elasticity b and the upper bound B are usually of order $\sim 10^0$, and the parameters a and c are in the order of $\sim 10^r$ and $\sim 10^k$, respectively. After all, the second term in the squared expression from (14) is in the order of $\sim 10^{r-k(b-1)-m} <<\sim 10^0$, as long as the order of magnitude of a is not comparatively very high.

5. Sensitivity Analysis of the Optimal Price, the Expected Profit, and the Variance of the Profit

5.1. Relationship between the optimal price and the risk parameter

We can analyze how, for a given stock factor, the optimal price changes as a function of the risk parameter λ. Let $\tilde{p}^*(\cdot, z)$ be a function of λ that denotes the optimal price given a stock factor z.

Lemma 2. *Given a stock factor z:*

a) If $\lambda \geq 0$ (risk-neutral and risk-averse cases), the optimal price is a nondecreasing function with respect to λ.

b) If $\lambda \in [\lambda_{min}, 0)$ (moderately risk-seeking cases), the optimal price is a nondecreasing function with respect to λ as long as $1 < b \leq 2$, or $b > 2$ and there are two real solutions to equation (5).

Corollary 1. *In the risk-averse case, the optimal price $\tilde{p}^*(\lambda, z)$ is always greater than or equal to the cost c since it follows from Lemma 2 that $\tilde{p}^*(\lambda, z) \geq \tilde{p}^*(0, z) \geq \tilde{p}^*(0, A) = bc/(b-1) > c$.*

The consequence of this lemma is that the optimal price for a given stock factor z, $\tilde{p}^*(\lambda, z)$ increases with the level of risk-aversion, whereas it decreases with the level of risk-seekingness. Although this result may seem counterintuitive at first sight, it is convenient to recall that one important characteristic of the multiplicative demand is that the price affects the demand uncertainty. More concisely, the variance of the demand is in this case decreasing with respect to the price, for $Var(D(p, \epsilon)) = Var(\epsilon)y(p)^2$ (Petruzzi and Dada, 1999). Therefore, when increasing λ in the risk-averse case, a price increase will reduce the expected demand $y(p)$, and this in turn will reduce the variance of the stochastic demand. Similarly, reducing λ in the risk-seeking case will increase the expected demand and induce an increment in the variance of the stochastic demand.

5.2. Relationship between the profit and the risk parameter

Let $\tilde{\Pi}^*(\lambda, z)$ be a random variable denoting the profit for a given stock factor z, as a function of the risk parameter λ.

Lemma 3. *The variance of the profit for a given stock factor z decreases as λ increases.*

Corollary 2. *As the newsvendor gets more risk-averse (risk-seeking), his optimal policy induces a smaller (greater) variance of the profit.*

Lemma 4. *The expected profit for a given stock factor z decreases as λ increases in the risk-averse case and decreases as λ decreases in the risk-seeking case.*

For illustration purposes, we analyze the example in §3.3 with $b = 3$ after embedding $p^*(\lambda, \cdot)$ in $P(\cdot, \cdot)$. Figure 4 shows the objective function $P^*(\lambda, \cdot)$, as well as $E(\tilde{\Pi}^*(\lambda, \cdot))$ and $Std(\tilde{\Pi}^*(\lambda, \cdot)) = \sqrt{Var(\tilde{\Pi}^*(\lambda, \cdot))}$, for different values of λ ranging from risk-seeking to risk-averse situations. All the curves represent values of λ above λ_{lim}. The behavior predicted by lemmas 3 and 4 can be observed in this figure: for a given stock factor z the variance of the profit decreases with the risk-aversion and increases with the risk-seekingness; in turn, the expected profit decreases with both risk-aversion and risk-seekingness. It is under the light of an example like this one where the power of a mean-variance analysis as a tool for decision-making can be seen: first we are able to come up with an array of optimal decisions as a function of our stance towards risk. The optimal value of the objective function itself is not significant; instead, it reveals an optimal stock factor and price that can be used for determining the best combination of expected profit and standard deviation of the profit for a particular risk tolerance. These are the true metrics when it comes to making a decision.

Secondly, the range of values for λ that are acceptable for every situation is given by the results that these values yield and the results derived from the sensitivity analysis previously shown: risk-averse decision-makers do not know at first what their tolerance to risk is in terms of λ but they know that there is a maximum standard deviation that is acceptable for them. Fine-tuning λ is thus a matter of finding the scenario that results in that maximum standard deviation. It follows from the sensitivity analysis that any λ greater than the value found will generate optimal pairs that guarantee lower standard deviations and this

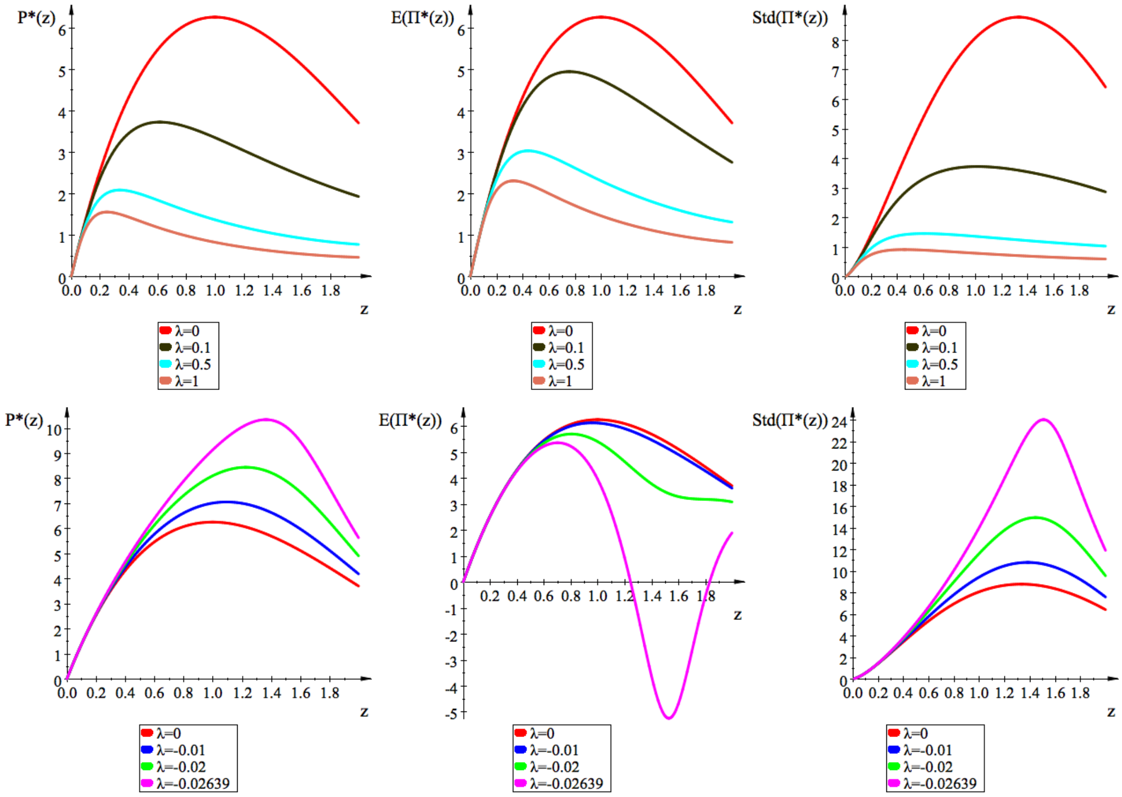


Figure 4: Objective function, expected profit, and standard deviation of the profit under different risk scenarios with $D(p, \epsilon) = 10^6 p^{-3}\epsilon$, $\epsilon \sim U[0.001, 1.999]$, $c = 100$

appreciation gives a range of values of λ. An analogous interpretation for risk-seeking individuals can be made using similar arguments in view of the results that stem from the sensitivity analysis.

Finally, Table 3 shows several numerical results for different values of λ ranging from risk-seeking cases to risk-averse cases. For these experiments we used a demand function $D(p, \epsilon) = 10^6 p^{-1.5}\epsilon$ with ϵ being distributed as three different distributions, namely, uniform, normal, and triangular. The cost of the commodity is assumed to be $c = 100$. For the range of values of λ used, $\lambda > \lambda_{lim}$ and therefore $p^*(z) > c$. Since $b = 1.5$, this optimal price can be calculated via the closed-form result shown in Table 1. Every scenario met condition a) from Theorem 4 and therefore the solution to the optimization problem is given by a unique pair $(p^*(z^*), z^*)$. Note that some risk-seeking scenarios even incur in expected loss profit in exchange for a higher standard deviation of the profit.

			$\lambda = -2.1E-04$					$\lambda = -1.2E-04$		
Distribution	p^*	z^*	P^*	$E[\Pi^*]$	$SD[\Pi^*]$	p^*	z^*	P^*	$E[\Pi^*]$	$SD[\Pi^*]$
Uniform [0.6, 1.4]	143.49	1.36	81,234.67	4,321.12	16,220.60	219.26	1.34	54,414.24	25,982.03	15,392.70
Normal $(1,0.25^2)$	140.70	1.43	84,732.33	-1,607.21	16,962.52	218.21	1.38	55,083.75	24,309	16,014.08
Triangular (0.3, 1.6, 1.1)	115.53	1.42	103,602.98	-21.677.25	19,882.49	193.26	1.39	61,731.77	19,749.96	18,704.23
			$\lambda = -3E-05$					$\lambda = 0$		
Distribution	p^*	z^*	P^*	$E[\Pi^*]$	$SD[\Pi^*]$	p^*	z^*	P^*	$E[\Pi^*]$	$SD[\Pi^*]$
Uniform [0.6, 1.4]	334,38	1.28	37,521.98	33,287.76	11,180.27	365.24	1.18	33,837.41	33,837.41	10,092.55
Normal $(1,0.25^2)$	330.45	1.25	37,317.70	33,067.93	11,902.05	359.59	1.15	33,646.23	33,646.23	10,137.24
Triangular (0.3, 1.6, 1.1)	325.86	1.28	38,164.38	32,672.02	13,530.66	367.91	1.18	33,432.89	33,432.89	11,484.84
			$\lambda = 3E-05$					$\lambda = 3E-04$		
Distribution	p^*	z^*	P^*	$E[\Pi^*]$	$SD[\Pi^*]$	p^*	z^*	P^*	$E[\Pi^*]$	$SD[\Pi^*]$
Uniform [0.6, 1.4]	366.83	1.04	31,475.43	33,220.21	7,626.22	333.95	0.77	26,986.03	28,622.65	3,525.68
Normal $(1,0.25^2)$	372.07	1.04	31,102.62	33,168.79	8,298.95	382.56	0.76	29,914.87	27,532.56	4,716.51
Triangular (0.3, 1.6, 1.1)	388.66	1.06	30,198.05	32,778.30	9,274.07	378.65	0.71	22,514.61	25,737.30	4,738.12

Table 3: Summary of results of the optimization problem for several random variables ($c = 100, y(p) = 10^6 p^{-1.5}$).

6. Conclusions

The results presented in this paper are oriented not only towards presenting conditions for the unimodality, but also towards giving those conditions a managerial appeal by writing them in terms of the LSR elasticity. This metric is closely related to a concept that is commonplace in industry: the level of service. One major problem of the multiplicative demand model is that, except for the risk-neutral case, a closed-form expression of the optimal price $p^*(\cdot)$ as a function of the price elasticity of the demand b does not exist. This is not the case in additive demand models, where the optimal price has a well-known formula that can be substituted in $P^*(\cdot)$, which then becomes fully characterized. Hence, it is more difficult to obtain constant bounds that guarantee the unimodality of the objective function in terms of the LSR elasticity. We acknowledge that, this being a function of only one variable, a numerical method should be able to return its maximum and a call to a plot function might suffice to identify it. However, we consider the results presented here as a first step towards attaining the goal of a more analytical set of results under which the unimodality of this function is guaranteed, on which we will focus our future research.

In spite of the hindrances posed by the isoelastic demand function, we are able to characterize the optimal price for a given stock factor and to determine if it is contained in the interval of prices under study $[c, \infty)$. More concisely, we find that this occurs if $\lambda \in [\lambda_{lim}, \infty)$ and therefore we focused our study on risk neutral, risk-averse, and moderately risk-seeking instances.

As a summary, the article contributes to the extension of the framework of LSR elasticity-related results to risk-neutral, risk-averse, and moderately risk-seeking instances. In risk-neutral cases, this contribution comes with the obtention of local and global optimality conditions for the unimodality of the profit function in the case of price-elastic goods, complementing the lower bound of $1/2$ given by Kocabıyıkoğlu and Popescu (2011) for the concavity of the problem with price-inelastic goods. In risk-sensitive settings we give conditions for the unimodality of any risk-averse and moderately risk-seeking instances. Two out of the three given bounds in Theorem 4 are not constant and require the evaluation of a function in the interval $[A, B]$. We aim at using these non-constant bounds as a first approach for finding better, more tractable, constant bounds . To conclude, a sensitivity analysis reveals that the expected profit decreases with the level of risk-aversion and the level of risk-seekingness, whereas the variance of the profit decreases with the level of risk-aversion and increases with the level of risk-seekingness.

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Appendix - Derivations of Common Mathematical Expressions

Derivation of $\mu(\cdot)$

Let $\mu(z) = \mathbb{E}(\min\{\epsilon, z\})$. Now, $\min\{\epsilon, z\}$ is the following random variable:

$$\min\{\epsilon, z\} = \begin{cases} u, \text{if } u \leq z, \\ z, \text{if } u > z. \end{cases}$$

Then, $\mu(z) = \mathbb{E}(\min\{\epsilon, z\}) = \int_A^z uf(u)du + \int_z^B zf(z)du$. The random variable ϵ has expected value $\mathbb{E}(\epsilon) = \int_A^z uf(u)du + \int_z^B uf(u)du = 1$, whence it follows that

$$\mu(z) = \int_z^B (z-u)f(z)du + 1.$$

Derivation of $\sigma^2(\cdot)$

Let $\sigma^2(z) = Var(\min\{\epsilon, z\}) = \mathbb{E}(\min\{\epsilon^2, z^2\}) - \mathbb{E}(\min\{\epsilon, z\})^2 = \mathbb{E}(\min\{\epsilon^2, z^2\}) - \mu^2(z)$. Now, $\min\{\epsilon^2, z^2\}$ is the following random variable:

$$\min\{\epsilon^2, z^2\} = \begin{cases} u^2, \text{if } u \leq z, \\ z^2, \text{if } u > z. \end{cases}$$

Then, $\mathbb{E}(\min\{\epsilon^2, z^2\}) = \int_A^z u^2 f(u)du + \int_z^B z^2 f(z)du = \mathbb{E}(\epsilon^2) - \int_z^B u^2 f(u)du + \int_z^B z^2 f(z)du$, where the last inequality follows from the fact that $\mathbb{E}(\epsilon^2) = \int_A^z u^2 f(u)du + \int_z^B u^2 f(u)du$. Since $\mathbb{E}(\epsilon^2) = Var(\epsilon) + \mathbb{E}(\epsilon)^2 = Var(\epsilon) + 1$, we obtain that $\mathbb{E}(\min\{\epsilon^2, z^2\}) = \int_z^B (z^2 - u^2)f(u)du + Var(\epsilon) + 1$. Finally, we conclude that

$$\sigma^2(z) = Var(\epsilon) + \int_z^B (z^2 - u^2)f(u)du - \mu^2(z) + 1.$$

Appendix - Proofs of Lemmas and Theorems

Proof of Theorem 1

Let us rewrite (2) as $\partial P(p)/\partial p = ap^{-b}M(p)$, where z is constant and has been omitted as a variable, and with $M(p) = 2\lambda\sigma^2(z)a(b-1)p^{-(b-1)} - (b-1)\mu(z) + bczp^{-1}$. We will show that the equation $\partial P(p)/\partial p = 0$ has only one solution. Indeed, since $\lambda > 0$ and $b > 1$, $M'(p) = -2(b-1)^2\lambda\sigma^2(z)ap^{-b} - bczp^{-2} < 0$. Moreover,

$M(0^+) = \infty$ and $M(\infty) = -(b-1)\mu(z) < 0$. Consequently, there is only one positive real solution p^* in the interval $(0, \infty)$ for each stock factor z.

On the other hand, $P(\cdot, z)$ is unimodal with respect to p for a stock factor z and $p^*(z)$ is a maximum. This follows because $M(p) > 0$ (i.e. $\partial P(p)/\partial p > 0$) in the interval $(0, p^*)$ and $M(p) < 0$ (i.e. $\partial P(p)/\partial p < 0$) in the interval (p^*, ∞).

Since $M(c) = 2\lambda\sigma^2(z)a(b-1)c^{-(b-1)} - (b-1)\mu(z) + bz = 2\lambda\sigma^2(z)a(b-1)c^{-(b-1)} + \mu(z) + b(z-\mu(z)) > 0$, it also follows that $p^*(z) > c$. Note that $(z-\mu(z))|_{z=A} = 0$, and it is an increasing function, hence $(z-\mu(z)) \geq 0$.

Proof of Theorem 2

(i) $\underline{1 < b \leq 2}$: The proof will proceed separately for two subintervals of b:

a) $\underline{1 < b < 2}$: Let us rewrite (2) as $\partial P(p)/\partial p = ap^{-b}M(p)$, where z is constant and has been omitted as a variable, and where $M(p) = M_0(p)p^{-1} - (b-1)\mu(z)$ and $M_0(p) = 2\lambda\sigma^2(z)a(b-1)p^{-(b-2)} + bcz$. We will show that the equation $\partial P(p)/\partial p = 0$ has only one solution. Indeed, since $\lambda < 0$ and $1 < b < 2$, $M_0'(p) = -2(b-2)(b-1)\lambda\sigma^2(z)ap^{-(b-1)} < 0$. Moreover, $M_0(0) = bcz > 0$ and $M_0(\infty) = -\infty$. Therefore, there exists only price $p_0 \in (0, \infty)$ such that $M_0(p) = 0$ and $M_0(p) > 0$ in $(0, p_0)$ and $M_0(p) < 0$ in (p_0, ∞). On the other hand, the sign of $M_0(\cdot)$ after p_0 makes $M(p) < 0$ in (p_0, ∞). In the interval $(0, p_0)$, $M(0^+) = \infty$, $M(p_0^-) = -(b-1)\mu(z) < 0$, and $M(\cdot)$ is strictly decreasing. Thus, there exists a price $p^* \in (0, p_0)$ such that $M(p^*) = 0$, with $M(p) > 0$ for $p \in (0, p^*)$ and $M(p) < 0$ for $p \in (p^*, \infty)$. Consequently, there is only one positive real solution p^* in the interval $(0, \infty)$ for each stock factor z, $P(\cdot, z)$ is unimodal with respect to p for a stock factor z and p^* is a maximum.

b) $\underline{b = 2}$: In this case (5) has the following unique solution: $p^*(z) = (2a\lambda\sigma^2(z) + 2cz)/\mu(z)$, and p^* in the interval $(0, \infty)$ for each stock factor $z \in [A,\ B]$ if and only if $\lambda \geq \lambda_{min}$. If $\lambda < \lambda_{min}$ there will be some values of z for which $p^*(z) < 0$ and others for which $p^*(z) \geq 0$.

Moreover, if we rewrite (2) as $\partial P(p)/\partial p = ap^{-3}M_1(p)$, where z is constant and $M_1(p) = 2\lambda\sigma^2(z)a - \mu(z)p + 2cz$, clearly, $\partial M_1(p)/\partial p = -\mu(z) < 0$, and $M_1(0) = 2\lambda\sigma^2(z)a + 2cz > 0$, when $\lambda \geq \lambda_{min}$. Since $M_1(p)$ is monotonically decreasing, $M_1(0) > 0$, and $M_1(\infty) = -\infty$, the result follows.

(ii) $\underline{b > 2}$: Equation (2) can be rewritten as as $\partial P(p)/\partial p = ap^{-(2b-1)}M_2(p)$, where z is constant and $M_2(p) = 2\lambda\sigma^2(z)a(b-1) - (b-1)\mu(z)p^{b-2} + bczp^{b-1}$, per *Descartes' Rule of Signs* (Ferreira and Machado, 2014, Anderson et al., 1998), $M_2(p) = 0$, thus $\partial P(p)/\partial p = 0$ has either two positive real roots or no positive real roots at all.

Proof of Lemma 1

Note that, for a constant stock factor z, the profit function $P(p,z)$ is still increasing when $p = c$, i.e.,

$$\left.\frac{\partial P(p,z)}{\partial p}\right|_{p=c} \quad = \quad ac^{-b}[2a(b-1)\sigma^2(z)\lambda c^{-(b-1)} - (b-1)\mu(z) + bz] > 0,$$

if and only if $\lambda > \lambda_{lim}$. Thus, for a stock factor z, for all $1 < b \leq 2$, the maximizer of $P(\cdot, z)$, is greater than c, i.e., $p^*(z) > c$. Moreover, in the case of $b > 2$, if there are two real roots, since $0 < p_1(z) < p_2(z)$, and the profit function is still increasing at c, the maximizer of $P(\cdot, z)$, is also greater than c, i.e., $p^*(z) = p_2(z) > c$.

Proof of Theorem 3

a) We first prove the local optimality condition. If the Hessian matrix of $P(\cdot,\cdot)$ is negative definite at $(p^*(z^*), z^*)$, then this point is a strict local maximum of $P(\cdot,\cdot)$ in $[A, B] \times [c, \infty)$. Given that $\partial^2 P/\partial z^2 = -ap^{-(b-1)}f(z) < 0$, per the *second derivative test* such a Hessian is negative definite as long as $\Delta(p^*(z^*), z^*) > 0$, where $\Delta(p,z) = \left(\partial^2 P/\partial p^2\right) \cdot \left(\partial^2 P/\partial z^2\right) - \left(\partial^2 P/\left(\partial p \partial z\right)\right)^2$. Using equations (3)-(4) with $\lambda = 0$, we can rewrite $\Delta(p,z)$ as

$$\Delta(p,z) = a^2 p^{-(2b+2)} \left(-pf(z)\left(b(b-1)\mu(z)p - (b+1)bcz\right) - \left(bc - (b-1)p\left(1 - F(z)\right)\right)^2\right). \qquad (15)$$

If we particularize for the set of prices that are optimal, $p^*(z)$, then

$$\Delta(p^*(z), z) = a^2 p^*(z)^{-(2b+2)} \left(p^*(z)bczf(z) - \left(bc - (b-1)p^*(z)\left(1 - F(z)\right)\right)^2\right), \qquad (16)$$

where we used the closed-form solution of the optimal price in the risk-neutral case, $p^*(z) = bcz/\left((b-1)\mu(z)\right)$, to simplify the right-hand side of the previous equation. Moreover, z^* satisfies the equation $F(z) = 1 - c/p^*(z)$ and thus we obtain $\Delta(p^*(z^*), z^*) = a^2 cp^*(z)^{-(2b+1)}\left(bz^* f(z^*) - (1 - F(z^*))\right)$. Therefore, and following the definition of LSR elasticity for isolastic demand as shown in equation (7), the condition $\Delta(p^*(z^*), z^*) > 0$ is possible if and only if $\tilde{\varepsilon}(z^*) > 1$. Proving that $(p^*(z^*), z^*)$ is a saddle point of $P(\cdot,\cdot)$ in $[c, \infty) \times [A, B]$ can be done analogously by imposing that $\Delta(p^*(z^*), z^*) < 0$.

b) To show that $P^*(p^*(\cdot), \cdot)$ is unimodal with respect to z, it is enough to check the behavior of $R(z)$, since $P^{*'}(z) = ap^*(z)^{-b}R(z)$, where $R(z) = [1 - F(z)]p^*(z) - c$, and $p^*(z) = bcz/[(b-1)\mu(z)]$. Clearly $P^{*'}(z) = 0$ if and only if $R(z) = 0$ if and only if

$$[1 - F(z)]bz = (b-1)\mu(z). \qquad (17)$$

Moreover, $R(A) = p^*(A) - c = c/(b-1) > 0$, and $R(B) = -c < 0$. Thus, by the continuity of $R(\cdot)$ there is at least one solution to the equation $P^{*'}(z) = 0$. In fact, $P^{*'}(\cdot)$ has only one root if and only if $R'(z)|_{R(z)=0} < 0$. If this happens, this root also represents the maximum of $P^*(\cdot)$, since $P^{*'}(A) > 0$, and $P^{*'}(B) < 0$. Note that,

$$R'(z) = -f(z)p^*(z) + [1 - F(z)]p^{*'}(z) \tag{18}$$

with

$$p^{*'}(z) = b(b-1)c[\mu(z) - (1 - F(z))z]/[(b-1)^2(\mu(z))^2]. \tag{19}$$

Substituting 17 and 19 in 18, we obtain $R'(z)|_{R(z)=0} = \frac{c}{(b-1)\mu(z)}[-bzf(z) + (1 - F(z)] < 0$ if and only if $\tilde{\varepsilon}(z) > 1$. Thus, $P^*(.)$ is unimodal with respect to z, and z^* is a maximum in the interval $[A, B]$.

c) Next, instead of the sequential decision process, i.e., optimizing $P(p, z)$ first with respect to $p \in [c, \infty)$ and then optimizing $P^*(p^*(z), z)$ with respect to $z \in [A, B]$, we consider optimizing $P(p, z)$ with respect to both z and p simultaneously. We prove the global simultaneous optimality condition using Theorem 1 in Christensen (2017), that refers to the *Poincaré-Hopf index theorem*. This theorem gives conditions for the uniqueness of the critical points of a multivariate function. Before we present the result, note that $P(p, z)$ is a continuous function and

$$\left.\frac{\partial P(p,z)}{\partial p}\right|_{p=c} > 0, \quad \left.\frac{\partial P(p,z)}{\partial p}\right|_{p=\infty} = 0_-,$$
$$\left.\frac{\partial P(p,z)}{\partial z}\right|_{z=A} > 0, \quad \left.\frac{\partial P(p,z)}{\partial z}\right|_{z=B} < 0.$$

Moreover, $\frac{\partial P(p,z)}{\partial p} < 0$ for $p > \frac{bcz}{(b-1)\mu(z)}$. Hence, there exists an $M > c$ such that $M = \max\{p : \frac{\partial P(p,z)}{\partial p} < 0\}$, thus implying that the maximizing point cannot be at the boundary, but an interior point of $[c, M] \times [A, B]$ (see Example 1 in Christensen (2017)). Theorem 1 states that $P(p, z)$ has a unique critical point in $[c, M] \times [A, B]$ which is a global maximum if and only if the determinant of the Hessian is positive at all critical points, i.e., $\Delta(p, z) > 0$ whenever $\frac{\partial P(p,z)}{\partial p} = 0$, $\frac{\partial P(p,z)}{\partial z} = 0$. Let p^{**} and z^{**} be the critical points that satisfy $bcz^{**} = (b-1)\mu(z^{**})p^{**}$ and $(1 - F(z^{**}))p^{**} = c$, then the condition becomes $\Delta(p^{**}, z^{**}) = a^2c(p^{**})^{-(2b+1)}(bz^{**}f(z^{**}) - (1 - F(z^{**}))) > 0$. But this is true since $\tilde{\varepsilon}(z) > 1$ for all $z \in [A, B]$. Hence, (p^{**}, z^{**}) is the unique solution that maximizes $P(p, z)$. Clearly, we can expand the domain of p to $[c, \infty)$, since $\left.\frac{\partial P(p,z)}{\partial p}\right|_{p=\infty} = 0_-$.

Proof of Theorem 4

The behavior of $P^{*'}(\cdot)$ is clearly given by that of $R(\cdot)$. At the limits of $[A, B]$ we have $R(A) = c/(b-1) > 0$, $R(B) = -c < 0$. This fact, along with the continuity of $R(\cdot)$, implies that there is at least one solution to the equation $P^{*'}(z) = 0$ (i.e. $R(z) = 0$ at least once). In fact, $P^{*'}(\cdot)$ has only one root if and only if $R'(z)|_{R(z)=0} < 0$. If this occurs, this root represents also a maximum of $P^{*}(\cdot)$, since $P^{*'}(A) > 0$ and $P^{*'}(B) < 0$. Note that,

$$\begin{aligned} R'(z) &= -f(z)p^{*}(z) + (1-F(z))p^{*'}(z) \\ &\quad -\lambda a p^{*}(z)^{-(b-1)}\left(p^{*}(z)\sigma^{2''}(z) - (b-2)p^{*'}(z)\sigma^{2'}(z)\right). \end{aligned} \tag{20}$$

In general, at the critical points of $P(\cdot)$ we have that

$$R(z) = 0 \quad \Longrightarrow \quad \lambda a p^{*}(z)^{-(b-2)}\sigma^{2'}(z) = (1-F(z))p^{*}(z) - c. \tag{21}$$

Substituting (21) in (20) and reordering terms, we obtain $R'(z)|_{R(z)=0} = -f(z)p^{*}(z) + (b-1)(1-F(z))p^{*'}(z) - (b-2)cp^{*'}(z)/p^{*}(z) - \lambda a p^{*}(z)^{-(b-2)}\sigma^{2''}(z)$. Dividing by $1-F(z)$ and using the equality $\sigma^{2''}(z) = 2(1-F(z))F(z) - 2f(z)(z-\mu(z))$ gives

$$\begin{aligned} \frac{R'(z)}{1-F(z)}|_{R(z)=0} &= \frac{p^{*'}(z)}{p^{*}(z)}\underbrace{\left((b-1)p^{*}(z) - \frac{(b-2)c}{1-F(z)}\right)}_{\text{Ⓑ}} + h(z)p^{*}(z)\underbrace{\left((z-\mu(z))2\lambda a p^{*}(z)^{-(b-1)} - 1\right)}_{\text{Ⓐ}} \\ &\quad -2\lambda a F(z)p^{*}(z)^{-(b-2)}. \end{aligned}$$

Ⓐ can be further particularized for $R(z) = 0$ using (21) to get $(z-\mu(z))2\lambda a p^{*}(z)^{-(b-1)} - 1 = -c/\left(p^{*}(z)(1-F(z))\right) < 0$. Furthermore, Ⓑ can be rewritten as $\left(c + (b-1)((1-F(z))p^{*}(z) - c)\right)/(1-F(z))$. Now, let $\eta(z) = (1-F(z))p^{*}(z) - c$. All in all, our condition for the negativity of $R'(z)|_{R(z)=0}$ results in $p^{*'}(z)\left((b-1)\eta(z) + c\right)/p^{*}(z) - F(z)\eta(z)/z - \mu(z) - h(z)c < 0$, whence we obtain:

$$h(z) \;>\; \left(\frac{p^{*'}(z)}{p^{*}(z)} + \left((b-1)\frac{p^{*'}(z)}{p^{*}(z)} - \frac{F(z)}{z-\mu(z)}\right)\frac{\eta(z)}{c}\right). \tag{22}$$

Multiplying both sides by bz and applying the definitions of (11) and (12) yields our first condition.

Now, it follows from (5) that

$$p^{*'}(z) \;=\; \frac{2\lambda a(b-1)\sigma^{2'}(z)p^{*}(z)^{-(b-2)} + bc - (b-1)\mu'(z)p^{*}(z)}{2\lambda a(b-1)(b-2)\sigma^{2}(z)p^{*}(z)^{-(b-1)} + (b-1)\mu(z)},$$

and

$$p^{*'}(z) \;=\; p^{*}(z)\frac{bc + \dfrac{\sigma^{2'}(z)}{\sigma^{2}(z)}\left((b-1)\mu(z)p^{*}(z) - bcz\right) - (b-1)\mu'(z)p^{*}(z)}{(b-1)^{2}\mu(z)p^{*}(z) - (b-2)bcz}, \tag{23}$$

after removing the explicit dependence on λ. Note that, because of (5), $(b-1)\mu(z)p^*(z) - bcz$ is positive in risk-averse cases ($\lambda > 0$), negative in risk-seeking cases ($\lambda < 0$), and 0 in risk-neutral cases ($\lambda = 0$). We can find $p^{*'}(z)/p^*(z)$ at those points where $R(z) = 0$. To see this, use $p^{*'}(z)$ as shown in (23) and particularize for those points by means of (21). The result can be manipulated to get

$$\begin{aligned}\left(p^{*'}(z)/p^*(z)\right)|_{R(z)=0} &= (c + (b-1)((1-F(z))p^*(z) - c)) \,/\, \left((b-1)^2\mu(z)p^*(z) - (b-2)bcz\right)\\ &= (c + (b-1)\eta(z)) \,/\, \left((b-1)^2\mu(z)p^*(z) - (b-2)bcz\right).\end{aligned}$$

This result, in conjunction with (22), yields

$$h(z) \;>\; \left(\frac{(c+(b-1)\eta(z))^2(1-F(z))(z-\mu(z))}{(b-1)^2\sigma^2(z)\eta(z) + bcz(1-F(z))(z-\mu(z))} - \frac{F(z)}{z-\mu(z)}\eta(z)\right)\frac{1}{c}. \tag{24}$$

Per (21), $\eta(z)|_{R(z)=0} \geq 0$ when $\lambda \geq 0$, and therefore we can bound (24) to get our second condition:

$$\begin{aligned}h(z) \;&>\; \left(\frac{(c+(b-1)\eta(z))^2(1-F(z))(z-\mu(z))}{(b-1)^2\sigma^2(z)\eta(z) + bcz(1-F(z))(z-\mu(z))} - \frac{F(z)}{z-\mu(z)}\eta(z)\right)\frac{1}{c}\\ &\leq\; \frac{(c+(b-1)\eta(z))^2(1-F(z))(z-\mu(z))}{(b-1)^2\sigma^2(z)\eta(z) + bcz(1-F(z))(z-\mu(z))}\frac{1}{c}\\ &\leq\; \frac{(c+(b-1)\eta(z))^2(1-F(z))(z-\mu(z))}{bcz(1-F(z))(z-\mu(z))}\frac{1}{c} = \frac{(c+(b-1)\eta(z))^2}{bc^2z}\\ &=\; \frac{((b-1)\eta(z)+c)^2}{bc^2z} \implies g(z) > \frac{1}{b}\left(\frac{(b-1)\eta(z)+c}{c}\right)^2.\end{aligned} \tag{25}$$

Using the equality $\tilde{\varepsilon} = bzh(z) = bg(z)$ we can write equations (24) and (25) as a function of the LSR elasticity, as shown in this theorem. For the last condition, assume that $b \geq 2$ and remember that the equation $R(z) = 0$ is equivalent to $\eta(z) = 2\lambda a(1-F(z))(z-\mu(z))/p^*(z)^{b-2}$. Our second condition can thus be written as $\tilde{\varepsilon}(z) > (1 + (b-1)\eta(z)/c)^2 \leq \left(1 + (b-1)2\lambda a(B-1)/c^{b-1}\right)^2$, and therefore we arrive to the our last lower bound:

$$\tilde{\varepsilon}(z) > \left(1 + \frac{2\lambda a(b-1)(B-1)}{c^{b-1}}\right)^2.$$

Proof of Lemma 2

Let $\lambda \in [\lambda_{min}, \infty)$ be a risk parameter for which an optimal price exists. This occurs either in risk-averse and risk-neutral instances (per Theorem 1) or in moderately risk-seeking instances when $1 < b \leq 2$ or when $b > 2$ and equation (5) has two real solutions (per Theorem 2). Let $\tilde{P}(\lambda, z, p)$ denote the objective function as a function of λ and p, and $g(\lambda, p) = \partial\tilde{P}(\lambda, p, z)/\partial p$. Per the *Implicit Function Theorem* we have that

$$\frac{d\tilde{p}^*(\lambda,z)}{d\lambda} = -\left.\frac{\dfrac{\partial g(\lambda,p)}{\partial\lambda}}{\dfrac{\partial g(\lambda,p)}{\partial p}}\right|_{p=\tilde{p}^*(\lambda,z)} = -\frac{2a^2(b-1)\sigma^2(z)\tilde{p}^*(\lambda,z)^{-2b+1}}{\left.\left(\dfrac{\partial^2\tilde{P}(\lambda,p,z)}{\partial p^2}\right)\right|_{p=\tilde{p}^*(\lambda,z)}} \geq 0,$$

since $\partial^2 \tilde{P}(\lambda, p, z)/\partial p^2 < 0$ when $p = \tilde{p}^*(\lambda, z)$ as was proved in theorems 1 and 2. Thus, the optimal price does not decrease with λ.

Proof of Lemma 3

From (1) and per Lemma 2 it follows that

$$\frac{d}{d\lambda} Var(\tilde{\Pi}^*(\lambda, z)) = \quad -a^2 \sigma^2(z) \tilde{p}^{*'}(\lambda, z) \frac{2b-2}{\tilde{p}^*(\lambda, z)^{2b-1}} \quad \leq 0.$$

Proof of Lemma 4

From (1) it follows that $E(\tilde{\Pi}^*(\lambda, z)) = a\mu(z)\tilde{p}^*(\lambda, z)^{-b+1} - cza\tilde{p}^*(\lambda, z)^{-b}$. Therefore $d\left(E(\tilde{\Pi}^*(\lambda, z))\right)/d\lambda = a\tilde{p}^{*'}(\lambda, z)\tilde{p}^*(\lambda, z)^{-b-1}\left(cbz - \mu(z)(b-1)\tilde{p}^*(\lambda, z)\right)$. Since the first factor is nonnegative, the sign of this derivative is given by that of the second factor shown above, which is nonpositive if and only if $\tilde{p}^*(\lambda, z) \geq cbz/\left((b-1)\mu(z)\right)$. Since, per (5), $\tilde{p}^*(0, z) = cbz/\left((b-1)\mu(z)\right)$ and given Lemma 2, we conclude that this factor is indeed nonpositive for $\lambda > 0$ and nonnegative for $\lambda < 0$.